LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 201 - CALCULUS III

Exam-I, Fall 2014

Duration: 75 minutes

INSTRUCTIONS: This exam consists of 7 pages and 5 problems. Check that none is missing. Answer the questions in the space provided for each problem; if more space is needed, you may use the back pages. To receive full credits, you have to justify your answers.

Student's Name: ________________

KEY

Grading scheme
(Keep it empty)

Question 1	/10
Question 2	/30
Question 3	/30
Question 4	/10
Question 5	/20
Total	/100

1. **[10 Points]** Use partial fractions and evaluate the following integral

$$\int \frac{4x^2 - 5x + 3}{x(x-1)^2}\, dx =$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{(A+B)x^2 + (-2A-B-C)x + A}{x(x-1)^2}$$

$A = 3, \; B = 1, \; C = 2$

$$\int \cdots dx = \int 3/x \, dx + \int \frac{dx}{x-1} + \int \frac{2}{(x-1)^2}\, dx$$

$$= 3 \ln|x| + \ln|x-1| - \frac{2}{(x-1)} + C$$

2. **[30 Points]** Calculate the following improper integrals

$$\int_1^e \frac{1}{x(\ln x)^2}\,dx =$$ there's a pole at $x=1$

$$\lim_{y\to 1^+} \int_y^e \frac{1}{x(\ln x)^2}\,dx$$

$$\int \frac{1}{x(\ln x)^2}\,dx = \int u^{-2}\,du = \frac{u^{-2+1}}{-2+1} = -\frac{1}{u} = -\frac{1}{\ln x}$$

$$\lim_{y\to 1^+} -\frac{1}{\ln x}\Big|_{x=y}^{x=e} = \lim_{y\to 1^+}\left\{-1+\frac{1}{\ln y}\right\} \longrightarrow +\infty$$

diverges

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx =$$ pole at 1

$$\lim_{y\to 1^-} \int_0^y \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int u\,du = u^2/2 = \frac{(\sin^{-1} x)^2}{2}$$

$$\lim_{y\to 1^+} \frac{1}{2}(\sin^{-1} y)^2 = \pi^2/8 .$$

$\int_{-\infty}^{\infty} \frac{1}{\cosh x}\, dx \quad = 2\int_0^{\infty} \frac{1}{\cosh(x)}\, dx$ $\qquad$ $\cosh(x)$ even

$$= 4\int_0^{\infty} \frac{1}{e^x+e^{-x}}\, dx = 4\int_0^{\infty} \frac{e^x}{(e^x)^2+1}\, dx$$

Now $\quad \int \frac{e^x}{(e^x)^2+1}\, dx = \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1} e^x + C$

So $\quad 4\int_0^{\infty} \frac{1}{e^x+e^{-x}}\, dx = 4\left[\lim_{b\to\infty} \tan^{-1} e^b - \tan^{-1} e^0\right] = 4\left[\pi/2 - \pi/4\right] = \pi$

3. **[30 Points]** Determine whether the following improper integrals converge or diverge. Justify your answer and precise the test your are using.

$$\int_0^{\infty} \frac{|\cos x|}{(e^x+1)^2}\, dx$$

DCT $\qquad 0 < |\cos x| < 1$, $\quad (e^x+1)^2 = e^{2x} + 1 + 2e^x > e^x$

$\therefore \quad \int_0^{\infty} \frac{|\cos x|}{(e^x+1)^2}\, dx < \int_0^{\infty} \frac{1}{e^x}\, dx \longrightarrow \lim_{b\to\infty}\left[e^{-b} - e^{-0}\right]$ converges

$\Rightarrow$ by DCT original diverges

$$\int_1^\infty \frac{\sqrt{x}}{\sqrt{x^4+4}}\,dx$$

$$f(x) = \frac{\sqrt{x}}{\sqrt{x^4+4}}$$

LCT with $g(x) = \sqrt{\frac{x}{x^4}} = \frac{1}{x^{1.5}}$

$\lim_{x\to\infty} \frac{f}{g} = 1 \Rightarrow$ original behaves like $\int g(x)\,dx$ which is p-integral $p = 1.5 > 1$

$\Rightarrow$ Original converges.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

$(\ln x) < x^{0.5}$ for $x \geq T$

$\Rightarrow \frac{(\ln x)}{x^2} < \frac{x^{0.5}}{x^2} = \frac{1}{x^{1.5}}$ $x \geq T$

$$\int_T^\infty \frac{(\ln x)}{x^2}dx \leq \underbrace{\int_T^\infty \frac{1}{x^{1.5}}\,dx}_{\text{Conv p-integral}}$$

$\Rightarrow$ By DCT, original converges.

4. **[10 Points]**

Find the value of a so that the given integral converges

$$\int_3^\infty \left(\frac{a}{x-1} + \frac{1}{x+1}\right) dx$$

type 1

$f(x)$

$$\int_3^\infty \frac{(a+1)x + (a-1)}{(x-1)(x+1)} dx$$

numerator behaves like $(a-1)$ = constant

$\boxed{\text{if } a = -1}$ $\{(a+1)=0\}$ $\Rightarrow$

$f(x) \sim \frac{(a-1)}{x^2} \Rightarrow \int_3^\infty \frac{(a-1)}{x^2} dx$ converges

Otherwise numerator behaves like $(a+1)x$

$\Rightarrow f \sim \frac{(a+1)}{x}$ but $\int \frac{a+1}{x} dx$ diverges

5. **[20 Points]** Determine if the following sequences converge or diverge.

(a) $a_n = \frac{(\ln n)^7}{\sqrt{n}}$. Hint: Use the sandwich theorem.

$0 \le (\ln n)^7 \le n^s \quad \forall \quad 0 < s < 0.5$

$\Rightarrow 0 \le \frac{(\ln n)^7}{\sqrt{n}} \le \frac{n^s}{n^{0.5}} = \frac{1}{n^{0.5-s}}$ positive

$\lim_{n \to \infty} \frac{1}{n^{0.5-s}} = 0$ since $0.5 - s > 0$

$\Rightarrow$ By sandwich $a_n \to 0$

(b) $b_n = \frac{n!}{6^n + 8^n} = \frac{1}{a_n}$

$$a_n = \frac{6^n + 8^n}{n!} = \frac{6^n}{n!} + \frac{8^n}{n!}$$

$$\lim_{n\to\infty} a_n = 0 + 0 \quad \text{form } \frac{x^n}{n!} \text{ from table}$$

$$\Rightarrow \lim_{n\to\infty} b_n = +\infty$$

(c) $c_n = \left(\frac{3}{n}\right)^{\frac{3}{n}}$

$$C_n = (3/n)^{3/n} = \frac{3^{3/n}}{(n^{1/n})^3}$$

$$\lim_{n\to\infty} 3^{3/n} = 1 \quad \text{since } 3/n \to 0$$

& 3 is fixed

$$\lim_{n\to\infty} n^{1/n} = 1$$

$$\Rightarrow C_n \longrightarrow 1$$